UDC 531.011

ON THE LOWER BOUNDS OF DISTANCE BETWEEN BODIES IN THE UNRESTRICTED THREE BODY PROBLEM

PMM Vol. 41, №4, 1977, pp. 609-617
V. G. GOLUBEV
(Moscow)
(Received September 14, 1976)

A method is proposed for the determination of the lower bound of distances of a pair of fixed bodies from a third body in the case of negative constant of the energy integral, when sufficient conditions of the Hill absolute stability of motion of the pair of bodies are assumed to be satisfied.

1. Statement of the problem. The unrestricted Newtonian problem of three points (bodies) P_1, P_2, and P_3 of mass m_1, m_2, and m_3, respectively, is considered in a system of coordinates whose origin lies at the baricenter (the center of mass of the set of three points). It is assumed that the constant vector $\mathbf{C}$ of the moment of momentum is nonzero, i.e. $C = |\mathbf{C}| > 0$. The case, most interesting from the application point of view, of $h < 0$, where h is the constant of the energy integral $T = U + h$ (T is the kinetic energy and U the force function) is analyzed.

When $C > 0$ triple collisions are impossible. According to Sundman (see, e.g., [1]) a mathematical solution of the problem exists for $-\infty < t < +\infty$ (t is the time), in spite of the possibility of dual collisions. Sundman's statement about the lower boundedness of the positive constant of perimeter $\Delta P_1P_2P_3$ when $C > 0$ is also important. It should be noted, however, that his theory is particularly complicated, while the estimates are nevertheless quite coarse; the latter may be due to the considerable generality of the case considered by him.

In the present paper the lower bounds for two (out of three) relative distances r_{jk} between the bodies, analogous to Sundman's lower bounds, are derived comparatively simply by using a supplementary assumption described below.

To explain the essence of that supplementary assumption we introduce the necessary notation and definitions. We present the energy integral in the form $T = U - h'$, where $h = -h > 0$ because of the assumption that $h < 0$. Note that $T > 0$, since owing to $C > 0$ it cannot vanish. Hence always $U > h'$. We introduce the relative masses of bodies $\mu_i = m_j / M$, $j = 1, 2, 3$, and $M = m_1 + m_2 + m_3$; evidently $0 < \mu_j < 1$ and $\mu_1 + \mu_2 + \mu_3 = 1$. Then the force function

$$U = fM^2\left(\frac{\mu_1\mu_2}{r_{12}} + \frac{\mu_1\mu_3}{r_{13}} + \frac{\mu_2\mu_3}{r_{23}}\right)$$

where f is Newton's gravitational constant. Since $U > h'$, we always have

$$\min_{j\neq K} r_{jk} < \frac{fM^2}{h'}(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3) \tag{1.1}$$

We separate one of the three pairs of bodies and denote its bodies by P_1 and P_2 such that $\mu_1 \geqslant \mu_2$, which can be evidently obtained by a suitable numbering of these. Let O_{12} be the center of mass P_1 and P_2. We set $r = O_{12}P_3$ and $\rho = r / r_{12}$, where r and ρ define the absolute and the relative distances of body P_3 from the pair P_1 and P_2 (only at the instant of collision of P_1 with P_2, $\rho = +\infty$). In what follows we shall need the following notation:

$$\lambda = \frac{\mu_1\mu_2}{\mu_1+\mu_2}, \quad \nu = \mu_3(\mu_1+\mu_2)$$
$$p = \frac{P_1O_{12}}{P_1P_2} = \frac{\mu_2}{\mu_1+\mu_2}, \quad q = \frac{O_{12}P_2}{P_1P_2} = \frac{\mu_1}{\mu_1+\mu_2}$$

Note that $0 < p \leqslant q < 1$ and $p + q = 1$.

The system moment of inertia I relative to the baricenter can now be expressed by the formula

$$I = \frac{m_1m_2}{m_1+m_2} r_{12}^2 + \frac{m_3(m_1+m_2)}{M} r^2 = Mr_{12}^2 i(\rho), \quad i(\rho) = \lambda + \nu\rho^2$$

It was shown in [2] and proved in [3] that when $r > qr_{12} \geqslant pr_{12}$, i.e. when $\rho > q \geqslant p$,

$$U \leqslant U_+ = \frac{fM^2}{r_{12}} u_+(\rho), \quad u_+(\rho) = \mu_1\mu_2 + \mu_3\left(\frac{\mu_1}{\rho+p} + \frac{\mu_2}{\rho-q}\right)$$

In other words, the quantity U for given $r_{12} > 0$ and $\rho > q \geqslant p$ attains its maximum when P_3 lies on the straight line P_1P_2 outside a like segment beyond point P_2.

On the basis of the remark about the numbering of bodies we attribute the following two definitions to the pair P_1 and P_2.

Definition 1.1 (see [2, 4, 5]). The motion of a specified pair of bodies P_1 and P_2 is called Hill stable if at all times $r_{12} < H$, where $H > 0$ is some constant.

Remark 1.1. The inequality (1.1) does not by itself imply the existence of even a single pair (out of three) whose motion is Hill stable.

Definition 1.2 (see [5]). The motion of the pair P_1 and P_2 is called absolutely Hill stable, if at all times $\rho \geqslant \rho_*$, where ρ_* is some constant such that $\rho_* > q \geqslant p$.

Remark 1.2. The absolute Hill stability implies simply the stability of motion of P_1 and P_2 and, also, the impossibility of collisions for other pairs of bodies (the first follows from the definitions and inequality (1.1) and the second, from the definition 1.2 and the impossibility of triple collisions when $C > 0$).

Below we assume that the motion of the specified pair P_1 and P_2 is absolutely Hill stable. This property of motions is ensured by the fulfilment of corresponding sufficient conditions. Before formulating these conditions we shall explain the prerequisites of these.

For $\rho > q \geqslant p$ from the inequality $IT \geqslant {}^1/_2C^2$, i.e. $I(U - h') \geqslant {}^1/_2C^2$, we have $I\,(U_+ - h') \geqslant {}^1/_2 C^2$, hence $IU_+^2 \geqslant 2h'C^2$.

We introduce the notation

$$S_+(\rho) = i(\rho)\,u_+^2(\rho), \quad s = \frac{2h'C^2}{f^2M^5}$$

(it is expedient to call the dimensionless constant s the Hill stability index). The last of inequalities can then be written as $S_+(\rho) \geqslant s$. Along $(q, +\infty)$ function $S_+(\rho)$ attains its absolute minimum $(S_+)_-$ at some point, whose position is determined by the known fifth power algebraic equation, to the left of which the function decreases and to the right of it it increases. Hence for $s > (S_+)_-$ equation $S_+(\rho) = s$ has two roots on $(q, +\infty)$. If ρ_* is the greater of these then $\rho \geqslant \rho_*$ is one of the solutions of the inequality $S_+(\rho) \geqslant s$.

Theorem (see [2, 3, 5]). Assuming that $s > (S_+)_-$ and ρ_* is the greater of the two roots of equation $S_+(\rho) = s$, and that at the initial instant t_0 $\rho(t_0) \geqslant \rho_*$, we have always $\rho \geqslant \rho_*$.

In what follows the conditions of the theorem are assumed to be satisfied without further stipulation.

Remark 1.3. Since ρ_* belongs to $(q, +\infty)$, $\rho_* > q$ and the inequality $\rho \geqslant \rho_*$ imply by virtue of definition 1.2 the absolute Hill stability of motions of P_1 and P_2.

Examples. Let us consider the problems of Sun (P_1) - Jupiter (P_2) - Saturn (P_3) and of Sun (P_1) - Earth (P_2) - Jupiter (P_3) with initial conditions for the epochs November 11, 1966 and January, 0, 1930, respectively (in each problem the "Solar system" is assumed to contain only two planets, and in the second problem the mass of Earth is taken as the sum of masses of the Earth and Moon).

On these assumptions in the first problem always $\rho > 1.319$, and in the second $\rho > 2.585$. Since in each case $0 < q < 1$ the inequalities defining ρ show that in the first problem the motion of the pair Sun - Jupiter and in the second that of Sun - Earth are absolutely Hill stable.

The following simple statement will be used repeatedly below.

Lemma 1.1. On assumptions indicated above always $r > 0$.

Proof. At the instant of collision between P_1 and P_2 when $r_{12} = 0$ clearly $r > 0$; since $r = 0$ implies a triple collision which is impossible because $C > 0$. If $r_{12} > 0$, from the inequality $r / r_{12} \geqslant \rho_* > 0$ we again have $r > 0$.

2. Derivation of the differential inequality for $r(t)$. The derivation for the quantity $r(t)$ of a differential inequality of the form $r^{\cdot\cdot} > \varphi(r)$ makes it possible to obtain for $r(t)$ a constant positive lower bound from which similar lower bounds follow for distances r_{13} and r_{23} Derivation of differential inequalities is more complicated than that of differential equations, since it is necessary to eliminate "extraneous" variables using the method of estimates. In that process the most laborious is the proof of Lemma 2.2. The approximate solution of inequality (2.16) is derived by an unusual method of successive approximations (corollaries 2.2 - 2.4).

Lemma 2.1. Let the motion of mass point J be defined in the inertial system of coordinates $Oxyz$ and $\mathbf{r} = \mathbf{OJ}$, with $r = |\mathbf{r}|$, $\mathbf{v}$ the velocity of the point, $v =$

$|\mathbf{v}|$, $\mathbf{a}$ is the acceleration, a_r is the projection of $\mathbf{a}$ on the $\mathbf{r}$-direction, $\mathbf{l} = [\mathbf{r} \times \mathbf{v}]$, and $l = |\mathbf{l}|$. Then

$$r^{\cdot\cdot} = a_r + l^2/r^3 \tag{2.1}$$

The proof is based on the identity

$$v^2 = r^{\cdot 2} + l^2/r^2 \tag{2.2}$$

Differentiating the relationship $r^2 = x^2 + y^2 + z^2$ twice we obtain

$$rr^{\cdot} = xx^{\cdot} + yy^{\cdot} + zz^{\cdot}$$
$$r^{\cdot 2} + rr^{\cdot\cdot} = v^2 + \mathbf{ra} = v^2 + r\,(\mathbf{ar}^{\circ}) = v^2 + ra_r$$

where $\mathbf{r}^{\circ}$ is the unit vector of vector $\mathbf{r}$, from which

$$r^{\cdot\cdot} = a_r + \frac{v^2 - r^{\cdot 2}}{r} \tag{2.3}$$

Now (2.1) follows from (2.3) and (2.2).

Corollary 2.1. If m is the mass of point J, $\mathbf{L} = [\mathbf{r} \times (m\mathbf{v})]$, and $L = |\mathbf{L}|$, then

$$r^{\cdot\cdot} = a_r + \frac{L^2}{m^2 r^3} \tag{2.4}$$

In fact, $\mathbf{L} = m\mathbf{l}$, $L = ml$, and $l = L\ /\ m$.

For obtaining the differential inequality for $r(t)$ it is expedient to pass to Jacobi coordinates in the three body problem. Let $\mathbf{r}_{12} = \{x_{12}, y_{12}, z_{12}\}$ be the vector which defines the relative position of P_2 to P_1 and $\mathbf{r} = \{x, y, z\}$ be the position of P_3 relative to O_{12}. The related Jacobi equations can be considered as equations that define the motion of two fictitious mass points J_{12} and J of mass $m_{12} = \lambda M$ and $m = \nu M$, respectively. The notation in Lemma 2.1 and Corollary 2.1 fit the second Jacobi point, and by analogy, the symbols of corresponding quantities for the first point, such as r_{12}, $r_{12}^{\cdot}$, $\mathbf{l}_{12}$, l_{12}, $\mathbf{L}_{12}$, and L_{12}, are natural.

It will be shown below that in Eq. (2.4) for J the inequalities $a_r < 0$ and $L > 0$ are (strictly) valid. Hence for the derivation of the inequality of the form $r^{\cdot\cdot} > \varphi(r)$ it remains to obtain for $a_r < 0$ the negative lower bound, and for $L > 0$ the positive lower bound both of which depend only on r.

Lemma 2.2. The estimate

$$a_r \geqslant -\frac{K}{r^2}, \quad K = fM\rho_*^{\,2}\left[\frac{p}{(\rho_* - q)^2} + \frac{q}{(\rho_* + p)^2}\right] \tag{2.5}$$

is valid.

Proof. In the notation used here the equations of motion of the second Jacobi point are of the form

$$x^{\cdot\cdot} = -fMx\left(\frac{p}{r_{23}^{\,3}} + \frac{q}{r_{13}^{\,3}}\right) + fMpqx_{12}\left(\frac{1}{r_{23}^{\,3}} - \frac{1}{r_{13}^{\,3}}\right) \tag{2.6}$$

(the right - hand side of equation for $y^{\cdot\cdot}$ is obtained from the above by substituting y for x and y_{12} for x_{12}, and similarly for $z^{\cdot\cdot}$).

We denote by ϑ the angle between vectors $\mathbf{r}_{12}$ and $\mathbf{r}$ $(0 \leqslant \vartheta \leqslant \pi)$ and set $\omega = \cos\vartheta$ $(-1 \leqslant \omega \leqslant 1)$. We have $a_r = \mathbf{a}\mathbf{r}^\circ = x^{\cdot\cdot}(x\ /\ r) + y^{\cdot\cdot}(y\ /\ r) + z^{\cdot\cdot}(z\ /\ r)$. From this and Eqs (2.6)

$$a_r = -fM\left[r\left(\frac{p}{r_{23}^3} + \frac{q}{r_{13}^3}\right) + pqr_{12}\omega\left(\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3}\right)\right]$$

Taking into account that from $\Delta P_1O_{12}P_3$ and $\Delta P_2O_{12}P_3$

$$r_{13}^2 = r^2 + p^2r_{12}^2 + 2pr_{12}r\omega, \quad r_{23}^2 = r^2 + q^2r_{12}^2 - 2qr_{12}r\omega \tag{2.7}$$

we can write

$$a_r = -fMG(\omega), \quad G(\omega) = \frac{p(r - qr_{12}\omega)}{r_{23}^3} + \frac{q(r + pr_{12}\omega)}{r_{13}^3} > 0 \tag{2.8}$$

Remark 2.1. We assume that $r_{12} \neq 0$ $(r_{12} > 0)$; it will be readily seen that at the instant of collision of P_1 and P_2 the inequality (2.5) is strict, since $\rho \to +\infty$ when $r_{12} \to 0$. That $G(\omega) > 0$ is implied by $r - qr_{12}\omega > 0$ and $r + pr_{12}\omega > 0$ by virtue of $\rho \geqslant \rho_* > q \geqslant p$; for example, $r - qr_{12}\omega \geqslant r - qr_{12} = r_{12}(\rho - q) \geqslant r_{12}(\rho_* - q) > 0$.

It remains to determine with the use of (2.8) the highest value of function $G(\omega)$ in $[-1, 1]$ for constant positive r_{12} and r, and for r_{13} and r_{23} that depend on ω in conformity with (2.7). From (2.7) we have

$$r_{13}'(\omega) = \frac{pr_{12}r}{r_{13}}, \quad r_{23}'(\omega) = -\frac{qr_{12}r}{r_{23}} \tag{2.9}$$

The differentiation of (2.8) with allowance for (2.9) yields

$$G'(\omega) = pqr_{12}\left[\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3} + 3r\left(\frac{r - qr_{12}\omega}{r_{23}^5} - \frac{r + pr_{12}\omega}{r_{13}^5}\right)\right] \tag{2.10}$$

From which, taking into account (2.7)

$$G'(\omega) = \frac{1}{2}pqr_{12}\left\{\left[\frac{1}{r_{23}^3} + \frac{3(r^2 - q^2r_{12}^2)}{r_{23}^5}\right] - \left[\frac{1}{r_{13}^3} + \frac{3(r^2 - p^2r_{12}^2)}{r_{13}^5}\right]\right\} \tag{2.11}$$

where, owing to $\rho > q \geqslant p$, the quantities in parentheses are positive and, what is important, independent of ω. When ω increases from -1 to 1, r_{13} increases and r_{23} decreases. Hence each fraction and, consequently, the whole expression in the first set of brackets in (2.11) increases, while the fractions and the complete expression in the second set of brackets decrease. Consequently $G'(\omega)$ increases, and the equation $G'(\omega) = 0$ can have only one root ω_* in the interval $(-1, 1)$, and $G'(\omega) < 0$ when $-1 \leqslant \omega < \omega_*$ and $G'(\omega) > 0$ when $\omega_* < \omega \leqslant 1$. This implies that $G(\omega)$ can have in $(-1,1)$ only one extremum which is a minimum. Hence the maximum value of function $G(\omega)$ along segment $[-1, 1]$ (which exists since that function is there continuous) is obtained at one of the ends of that segment.

We have $r_{13}(\pm 1) = r \pm pr_{12}$ and $r_{23}(\pm 1) = r \mp qr_{12}$. Hence by formula (2.8)

$$G(\pm 1) = \frac{p}{(r \mp qr_{12})^2} + \frac{q}{(r \pm pr_{12})^2}$$

$$G(1) - G(-1) = 4pqr_{12}r\left[\frac{1}{(r^2 - q^2r_{12}^2)^2} - \frac{1}{(r^2 - p^2r_{12}^2)^2}\right]$$

But $0 < r^2 - q^2 r_{12}^2 \leqslant r^2 - p^2 r_{12}^2$ (since $p \leqslant q$), hence $G(1) - G(-1) \geqslant 0$. Consequently $G(\omega) \leqslant G(1)$ and according to (2.8)

$$a_r \geqslant -fMG(1) = -\frac{fM}{r^2} g(\rho), \quad g(\rho) = \rho^2 \left[\frac{p}{(\rho - q)^2} + \frac{q}{(\rho + p)^2}\right] \qquad (2.12)$$

But in $(q; +\infty)$

$$g'(\rho) = 2pq\rho \left[\frac{1}{(\rho + p)^3} - \frac{1}{(\rho - q)^3}\right] < 0$$

which implies that function $g(\rho)$ in $(q, +\infty)$ and in particular in $[\rho_*, +\infty)$ decreases. Hence $a_r \geqslant -fMg(\rho_*)/r^2$ which yields (2.5).

Lemma 2.3. Let

$$P = \sqrt{2\nu Mh'}, \quad Q = fM^3 \sqrt{\lambda\nu} u_+(\rho_*) \qquad (2.13)$$

Then

$$0 < Q/P < C \qquad (2.14)$$

Proof. Since P and Q are positive constants, it is necessary to prove only the inequality $Q/P < C$. First we shall prove that by the basic theorem (see Sect. 1)

$$u_+(\rho_*) < \sqrt{s/\lambda} \qquad (2.15)$$

In fact, $s = S_+(\rho_*) = i(\rho_*) u_+^2(\rho_*) = (\lambda + \nu\rho_*^2) u_+^2(\rho_*) > \lambda u_+^2(\rho_*)$ from which follows (2.15). We recall that $s = 2h'C^2/(f^2M^5)$. Then by formulas (2.13) and inequality (2.15)

$$\frac{Q}{P} = \frac{fM^2\sqrt{M}}{\sqrt{2h'}} \sqrt{\lambda}\, u_+(\rho_*) = \frac{\sqrt{\lambda}\, u_+(\rho_*)}{\sqrt{s}} C < C$$

We pass to the problem of determining the lower bound of quantity L (depending on r) (see the text between Lemmas 2.1 and 2.2 above).

Lemma 2.4. Let $L < C$. Then

$$(C - L)^2 (L^2 + P^2 r^2) \leqslant Q^2 r^2 \qquad (2.16)$$

Proof. Jacobi equations (the motion of two fictitious points) contain integrals of energy and areas which are obtained from the integrals in baricentric coordinates by transformation to Jacobi variables, with the constants h' and C retaining their values. Using the energy integral we obtain the following obvious expression for the doubled kinetic energy of the system of two Jacobi points:

$$2(U - h') = \lambda M \left(r_{12}^{\cdot 2} + \frac{l_{12}^2}{r_{12}^2}\right) + \nu M \left(r^{\cdot 2} + \frac{l^2}{r^2}\right)$$

By virtue of $l_{12} = L_{12}/(\lambda M)$ and $l = L/(\nu M)$ we have

$$2(U_+ - h') \geqslant \frac{L_{12}^2}{\lambda M r_{12}^2} + \frac{L^2}{\nu M r^2} \qquad (2.17)$$

$$2\left[\frac{fM^2}{r} \rho u_+(\rho) - h'\right] \geqslant \frac{\rho^2 L_{12}^2}{\lambda M r^2} + \frac{L^2}{\nu M r^2}$$

where allowance is made for $U \leqslant U_+ = fM^2 r_{12}^{-1} u_+(\rho) = fM^2 r^{-1} \rho u_+(\rho)$ because $\rho \geqslant$

$\rho_* > q$. In $(q, +\infty)$ and in particular in $[\rho_*, +\infty)$ function $u_+(\rho)$ decreases so that $u_+(\rho) \leqslant u_+(\rho_*)$ when $\rho \geqslant \rho_*$. Strengthening the second of inequalities (2.17) and then multiplying it by $\lambda \nu M r^2 > 0$, we obtain

$$2\lambda\nu Mr\,[fM^2\rho u_+(\rho_*) - h'r] \geqslant \nu\rho^2 L_{12}^2 + \lambda L^2 \tag{2.18}$$

From the integral of areas $\mathbf{L}_{12} + \mathbf{L} = \mathbf{C}$ follows that $C = |\mathbf{C}| = |\mathbf{L}_{12} + \mathbf{L}| \leqslant |\mathbf{L}_{12}| + |\mathbf{L}| = L_{12} + L$, i.e.

$$L_{12} + L \geqslant C \tag{2.19}$$

By stipulation $L < C$, which with (2.19) yields

$$L_{12}^2 \geqslant (C - L)^2 > 0 \tag{2.20}$$

From (2.18) and (2.20)

$$\nu (C - L)^2\rho^2 - 2\lambda\nu fM^3 u_+(\rho_*)\, r\rho + \lambda\,(L^2 + 2\nu Mh'r^2) \leqslant 0$$

Such quadratic inequality with respect to ρ is only possible when the discriminant of its left-hand side is nonnegative. Hence

$$\lambda^2\nu^2 f^2 M^6 u_+^2(\rho_*)\, r^2 - \lambda\nu\,(C - L)^2\,(L^2 + 2\nu Mh'r^2) \geqslant 0$$

The last inequality, after its division by $\lambda\nu$ and with allowance for the notation in (2.13), yields (2.16).

Corollary 2.2. The inequality $L > 0$ is strictly true.

Let on the contrary $L = 0 < C$. Then from (2.16) $C^2P^2r^2 \leqslant Q^2r^2$. Since by Lemma 1.1 $r > 0$, hence $C^2P^2 \leqslant Q^2$ and $Q / P \geqslant C$ which contradicts (2.14). We have to assume that $L > 0$.

Corollary 2.3. Let

$$\Lambda = C - Q/P \quad (0 < \Lambda < C) \tag{2.21}$$

(in establishing the inequalities for Λ allowance is made for (2.14)). The strict inequality $L > \Lambda$ is true.

In fact, if $L \geqslant C$, then $L > \Lambda$ since $C > \Lambda$. Let now $L < C$. From (2.16), taking into account that $L > 0$, we obtain $(C - L)^2P^2r^2 < Q^2r^2$, $(C - L)^2P^2 < Q^2$, $0 < C - L < Q / P$, and $L > C - Q / P = \Lambda$.

Corollary 2.4. The following strict inequality is valid:

$$L > C - \frac{Qr}{\sqrt{\Lambda^2 + P^2r^2}} \tag{2.22}$$

which is obvious when $L \geqslant C$. Let $L < C$, then from (2.16) we have $(C - L)^2 < Q^2r^2 / (\Lambda^2 + P^2r^2)$, since $L > \Lambda$. Extracting the square root from both sides of the last inequality and taking into consideration that $C - L > 0$, we again have (2.22).

From (2.4), (2.5), and (2.22) with allowance for $m = \nu M$ we immediately obtain the following lemma.

Lemma 2.5. The following strict inequality is true:

$$r^{\cdot\cdot} > \varphi(r), \quad \varphi(r) = \frac{1}{\nu^2M^2r^3}\left(C - \frac{Qr}{\sqrt{\Lambda^2 + P^2r^2}}\right)^2 - \frac{K}{r^2} \tag{2.23}$$

where the constants K, P, Q, and Λ are defined by formulas (2.5), (2.13), and (2.21).

3. Use of the differential inequality for $r(t)$.

Lemma 3.1. There exists in the interval $0 < r < +\infty$ an r_0 such that $\varphi(r) > 0$ when $0 < r < r_0$, $\varphi(r_0) = 0$, and $\varphi(r) < 0$ when $r > r_0$.

Proof. We have

$$\varphi(r) = \frac{1}{\nu^2 M^2} \frac{\varphi_1(r)}{r^3}, \quad \varphi_1(r) = \left(C - \frac{Qr}{\sqrt{\Lambda^2 + P^2 r^2}}\right)^2 - \nu^2 M^2 K r \tag{3.1}$$

It will be readily seen that in $(0, +\infty)$, $\varphi_1(r)$ is a decreasing function, to wit, it decreases from $C^2 > 0$ to $-\infty$ when r varies from 0 to $+\infty$. This remark and the first of formulas (3.1) make the lemma obvious.

Theorem 3.1. Any maximum of function $r(t)$ is strictly greater than r_0.

Proof. Note that according to Sundman function $r(t)$ is fairly smooth in spite of possible instants of collision between P_1 and P_2. Let function $r(t)$ reach its maximum r_+ at some instant t_1 with $r_+ \leqslant r_0$ contrary to the statement of the theorem. Then $\dot r(t_1) = 0$ and $\ddot r(t_1) \leqslant 0$. But by Lemma 3.1 $\varphi(r_+) \geqslant 0$ and according to (2.23) $\ddot r(t_1) > 0$. This contradiction proves the theorem.

Examples. In the problems Sun (P_1) -Jupiter (P_2) -Saturn (P_3) and Sun (P_1) - Earth (P_2) -Jupiter (P_3) any maximum of function $r(t)$, i.e. the distances of P_3 from the center of mass of P_1 and P_2 are greater than 6.412 a.u. and 5.045 a.u., respectively (a.u. denotes the astronomical unit).

Let us now consider in $(0, +\infty)$ the function

$$\psi(r) = \frac{C^2}{\nu^2 M^2 r^2} + \frac{2Q^2}{\nu^2 M^2 \Lambda^2} \ln \frac{\sqrt{\Lambda^2 + P^2 r^2}}{Pr} - \frac{2K}{r} - \frac{4CPQ}{\nu^2 M^2 \Lambda^2}\left(\frac{\sqrt{\Lambda^2 + P^2 r^2}}{Pr} - 1\right) \tag{3.2}$$

Obviously

$$\lim_{r \to +0} \psi(r) = +\infty, \quad \lim_{r \to +\infty} \psi(r) = 0 \tag{3.3}$$

It is not difficult to verify that

$$\psi'(r) = -2\varphi(r) \tag{3.4}$$

Lemma 3.2. In the interval $0 < r < r_0$ function $\psi(r)$ decreases and in the interval $r_0 < r < +\infty$ it increases, reaching its negative absolute minimum at point r_0. The equation $\psi(r) = 0$ has a single root r_* which belongs to the interval $(0, r_0)$, and the function $\psi(r) > 0$ when $0 < r < r_*$, and $\psi(r) < 0$ when $r_* < r < +\infty$.

Lemma 3.3. The quantity

$$R = \dot r^2 + \psi(r) \tag{3.5}$$

varies in conformity with r, it increases with increasing r and diminishes with decreasing r, and attains the same kind of extrema as r.

Lemma 3.2 follows from (3.4), Lemma 3.1 and (3.3), while Lemma 3.3 follows from (3.5), (3.4), and (2.23).

Theorem 3.2. Let at the initial instant t_0 $r(t_0) > r_*$ and $R(t_0) \leqslant 0$. Then always $r(t) > r_*$.

Proof. For the initial moment the statement is true by stipulation. Let us prove its validity in the interval $t_0 < t < +\infty$ by assuming the contrary. Then, owing to the continuity of $r(t)$ there must exist an instant $t_1 > t_0$ such that $r(t_1) = r_*$, and $r(t) > r_*$ when $t_0 \leqslant t < t_1$. The definition of derivative (if $\Delta t < 0$ is assumed) implies that $r^{\cdot}(t) < 0$ (2.23) implies that $r^{\cdot\cdot}(t_1) > \varphi(r_*) > 0$, since $0 < r_* < r_0$. These two facts imply the existence of a $\delta > 0$ such that $r^{\cdot}(t) < 0$ when $t_1 - \delta < t < t_1$. There are only two possibilities: a) that $r^{\cdot}(t_1) \leqslant 0$ when $-\infty < t < t_1$ (in particular when $t_0 < t < t_1$), and b) that there exists an instant $t_2 < t_1$ such that $r^{\cdot}(t_2) = 0$ and $r^{\cdot}(t) < 0$ when $t_2 < t < t_1$.

Case a). By Lemma 3.3 $R(t_1) < R(t_0) \leqslant 0$, i.e. $R(t_1) < 0$ But by (3.5) $R(t_1) = r^{\cdot 2}(t_1) + \psi(r_*) = r^{\cdot 2}(t_1) \geqslant 0$ (a contradiction).

Case b). By Lemma 3.3 $R(t_1) < R(t_2) = r^{\cdot 2}(t_2) + \psi[r(t_2)] = \psi[r(t_2)] < 0$ (since $r(t_2) > r(t_1) = r_*$), i.e. $R(t_1) < 0$. But by (3.5) we have again $R(t_1) \geqslant 0$ (contradiction).

These two contradictions prove the validity of the theorem for $(t_0, +\infty)$. The proof for $(-\infty, t_0)$ is similar.

Corollary 3.1. When the conditions of the theorem in Sect. 1 and of Theorem 3.2 are satisfied, then always

$$r_{13} > \left(1 - \frac{p}{\rho_*}\right) r_*, \quad r_{23} > \left(1 - \frac{q}{\rho_*}\right) r_* \tag{3.6}$$

First we point out that the multipliers in parentheses in (3.6) are strictly positive because $\rho_* > q \geqslant p$. Estimates (3.6) follow from the inequalities for $\Delta P_1 O_{12} P_3$ and $\Delta P_2 O_{12} P_3$

$$r_{13} \geqslant r - p r_{12} = (1 - p/\rho) r > (1 - p/\rho_*) r_*$$
$$r_{23} \geqslant r - q r_{12} = (1 - q/\rho) r > (1 - q/\rho_*) r_*$$

Examples. In the problem Sun (P_1) - Jupiter (P_2) - Saturn (P_3) always $r > 3.756$ a.u. (any maximum of $r(t) > 6.412$ a.u.), $r_{13} > 3.753$ a.u., and $r_{23} > 0{,}912$ a.u. In the problem Sun (P_1) - Earth (P_2) - Jupiter (P_3) always $r > 2.532$ a.u. (any maximum of $r(t) > 5.045$ a.u.), $r_{13} > 2.532$ a.u., and $r_{23} > 1.553$ a.u.

REFERENCES

1. Duboshin, G. N., Celestial Mechanics. Analytical and Qualitative Methods. Moscow, "Nauka", 1964.

2. Golubev, V. G., On the Hill stability in the unrestricted three body problem. Dokl. Akad. Nauk SSSR, Vol. 180, №2, 1968.

3. Golubev, V. G., Qualitative analysis of certain properties of motion in the three body problem. Tr. Mosk. Energ. Inst. Ser. Matem. №89, 1971.

4. Golubev, V. G., Certain corollaries of classical integrals in the three body problem. Tr. Mosk. Energ. Inst. Ser. Matem. №89, 1971.

5. Golubev, V. G., On sufficient conditions of absolute Hill stability in the unrestricted three body problem. Tr. Mosk. Energ. Inst., №260, 1975.

Translated by J. J. D.